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The minimum cost shortest-path tree game

F.R. Fernández · J. Puerto

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Abstract A minimum cost shortest-path tree is a tree that connects the source with every node of the network by a shortest path such that the sum of the cost (as a proxy for length) of all arcs is minimum.

In this paper, we adapt the algorithm of Hansen and Zheng (Discrete Appl. Math. 65:275–284, 1996) to the case of acyclic directed graphs to find a minimum cost shortest-path tree in order to be applied to the cost allocation problem associated with a cooperative minimum cost shortest-path tree game. In addition, we analyze a non-cooperative game based on the connection problem that arises in the above situation. We prove that the cost allocation given by an 'à la' Bird rule provides a core solution in the former game and that the strategies that induce those payoffs in the latter game are Nash equilibrium.

Keywords Operations research games · Core solution · Nash equilibrium

1 Introduction

The concept of spanning tree is of major importance when constructing network models that connect a set of users to a source using the smallest amount of resources. When the total cost to connect all nodes (including the source) is minimized, the focus is on the identification of minimum cost spanning trees. However, when the objective is to find the set of arcs connecting all the nodes such that the sum of the arc costs (as a proxy for length) from the source to each node is minimized, we consider a minimum cost shortest-path tree. The study of both types of trees has been an important area of research. A number of efficient algorithms have been developed to construct minimum cost trees (see Wu and Chao 2004 for a survey). Both types of trees have been widely used by communication companies in circuit design, cable T.V. networks, etc. (see Bharath-Kumar and Jaffe 1983; Borm et al. 2001; Marín 2007; Wu and Chao 2004).

F.R. Fernández · J. Puerto (✉)
Institute of Mathematics (IMUS), Seville University, Seville, Spain
e-mail: puerto@us.es

F.R. Fernández
e-mail: fernande@us.es

We are interested in the minimum cost shortest-path tree problem (see Hansen and Zheng 1996). A minimum cost shortest-path tree is a tree that connects the source to every node of the network by a shortest path such that the sum of the cost of all arcs is minimum. Shortest-path trees from the source to each node may not be unique. Further, minimum cost shortest-path trees may not be unique.

For routing problems, all shortest-path trees from the origin to any node are equivalent. However, trees with a minimum total length are interesting to be considered for building problems.

This class of problems appears in location models when (in a network) we want to connect the users to the distribution center at the minimum cost. Besides, the minimum cost shortest-path tree also provides a good approximation to the minimum routing cost spanning tree problem using the median node of the network as a source node for the tree (see Wu and Chao 2004).

Apart from the design of minimum cost networks, other interesting issues usually arise with these problems. For example, we may consider the problem of allocating the cost of the networks among the users, then the situation can be modeled as a game (see Borm et al. 2001; Curiel 1997; Fragnelli et al. 2000; Granot and Huberman 1981; Voorneveld and Grahn 2002).

From a game theory perspective we may explore two situations for the allocation of costs in spanning trees. In a cooperative environment, all the agents cooperate to develop a stable allocation of costs. However, the agents can act in a non-cooperative way. In this case, the agents will adopt strategies that are crucial to the outcome of the game.

The rest of the paper is organized as follows. In the next section, we adapt the algorithm in Hansen and Zheng (1996) to find the minimum cost shortest-path tree (MCSPT) in a connected acyclic graph. In Sect. 3, we study a cooperative game to allocate the total cost of the tree among the agents. Finally, in Sect. 4 we investigate the allocation of the cost of the tree among the agents using a non-cooperative multi-stage game. The payoffs for the agents given by the strategies of the non-cooperative game coincide with the cost allocations in the cooperative game, so the non-cooperative solutions can be interpreted as implementations of the cooperative solutions.

2 The minimum cost shortest-path tree problem

Let $G=(N_0,A)$ be an acyclic digraph that has at least one spanning tree as a subgraph, $N_0=N\cup\{0\}$, where 0 denotes the source node (or the common supplier), $N=\{1,\dots,n\}$ the rest of the nodes, and A the set of arcs, $A\subset N_0\times N_0$.

We denote by l_{ij} the cost (which is a proxy for length) of arc $(i,j)\in A$. We assume $l_{ij}\geq 0$ for all $(i,j)\in A$.

A spanning tree $T(S)$, $S\subset N_0$, rooted at 0 is an acyclic graph with a unique path from 0 to every node j, $j\in S$. For the sake of simplicity let $A(S)$ be its set of arcs. Therefore, $T(S)=(S\cup\{0\},A(S))$. A spanning tree of a connected graph G is a spanning tree $T(N)$ containing all the nodes N_0 in G.

A shortest-path tree T_S, $S\subset N_0$, rooted at 0 is a spanning tree such that the length, $\pi^*(j)$, of the path from 0 to j, $\forall j\in S$, is the shortest length among all the paths from 0 to j in G. Let A_S be the set of its arcs. So, $T_S=(S\cup\{0\},A_S)$.

The cost, $c(T_S)$, associated with a tree T_S, is the sum of the costs of all its arcs:

$$c(T_S)=\sum_{(i,j)\in A_S} l_{ij}.$$

A *minimum cost shortest-path tree* (MCSPT) of the graph G is a shortest-path tree T_N for G such that $c(T_N)$ is minimum.

In Hansen and Zheng (1996) one can find an algorithm to search for a minimum cost shortest-path tree. Our goal is to adapt that algorithm, based on a recursion '*à la*' Bellman, that will be useful later on when analyzing the cooperative game on Sect. 3. We have introduced this adaptation for the sake of completeness, because our version of the algorithm will be essential later on to understand the construction. The process takes n steps to determine the total cost associated with an optimal tree that has the shortest-path from node 0 to every node of the graph. This results in a sequence of optimal subtrees with costs in nondecreasing order.

For the sake of presentation, let $\Gamma_{i_k}^{-1} = \{i \in N_0 : (i, i_k) \in A\}$, i.e. the set of predecessors of i_k (see Gondran and Minoux 1984).

We number the nodes of the graph with a numbering compatible with the precedence relationship of A. (Any numbering compatible with the precedence relationship would be valid.) This allows us to construct the following sequence:

$$S_0 \subset S_1 \subset \cdots \subset S_k \subset \cdots \subset S_n. \tag{2.1}$$

$S_0 = \{0\}$, $S_1 = \{0, i_1\}$, where i_1 is a node whose unique predecessor is the origin, $S_2 = S_1 \cup \{i_2\}$, where i_2 is a node for which all its predecessors are in S_1, and, in general, $S_k = S_{k-1} \cup \{i_k\}$, where i_k is a node for which all its predecessors lie in S_{k-1}, and so on until $S_n = \{0, 1, \dots, n\}$. The reader may note that there are many sequences satisfying the above construction, depending on the choice of the node at each step. The validity of our approach is independent of the chosen sequence.

Using a sequence introduced in (2.1), we define the optimality relation as:

$$\pi^*(0) = 0; \qquad T_{S_0} = (S_0, \emptyset) \tag{2.2}$$

and for any $k > 0$:

$$\pi^*(i_k) = \min_{i \in \Gamma_{i_k}^{-1}} \{l_{ii_k} + \pi^*(i)\}; \qquad T_{S_k} = (S_k, A_{S_{k-1}} \cup \{(i_k^*, i_k)\}) \tag{2.3}$$

where i_k^* is a single node, that can be arbitrarily chosen among all the $\widehat{i_k^*} \in \arg\min_{i \in \Gamma_{i_k}^{-1}} \{l_{ii_k} + \pi^*(i)\}$. Let us define $A(i_k) = \{(\widehat{i_k^*}, i_k) : \widehat{i_k^*} \in \arg\min_{i \in \Gamma_{i_k}^{-1}} \{l_{ii_k} + \pi^*(i)\}\}$ being the set of arcs verifying $l_{\widehat{i_k^*}, i_k} = \pi^*(i_k) - \pi^*(\widehat{i_k^*})$.

It is well-known (see Gondran and Minoux 1984; Wu and Chao 2004), that T_{S_n} is a shortest-path tree on G, and $\pi^*(k)$ is the length of any shortest-path from the origin 0 to the node k, $k = 1, \dots, n$.

Moreover, we prove that if we choose the node i_k^* satisfying

$$l_{i_k^* i_k} \le l_{ii_k}, \quad \forall i \in \arg\min_{i \in \Gamma_{i_k}^{-1}} \{l_{ii_k} + \pi^*(i)\} \tag{2.4}$$

the above procedure also determines a minimum cost shortest-path tree of G. Let $A_{i_k}^*$ be the set of arcs in $A(i_k)$ that satisfy (2.4).

Theorem 2.1 *The sequence of arcs* $(i_k^*, i_k), k = 1, \dots, n$, *determined by* (2.2), (2.3) *and* (2.4) *results in a minimum cost shortest-path tree* T_{S_n} *of* G.

Proof Let T_{S_n} be a tree determined by the above procedure, and let us assume that there exists another shortest-path tree, T^*, for which $c(T^*) < c(T_{S_n})$.

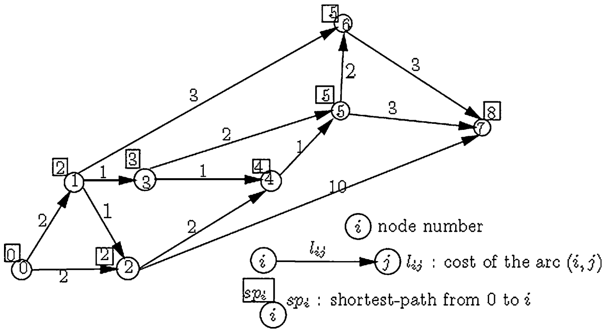

Fig. 1 Distribution graph and shortest-paths on the graph

To each node of any tree of the graph G we associate a label given by the length of the arc incident to it in the unique path from the origen to it on the tree. Then, since $c(T^*) < c(T_{S_n})$ it must exist, at least, one node i with a label, l_{j^*i}, given by T^*, that has smaller label than the one given by T_{S_n}, $l_{\widehat{j}i}$, i.e. $l_{j^*i} < l_{\widehat{j}i}$.

As T^* is a shortest-path tree, it can be obtained using the principle of optimality by a recursive chain of sets S_l^* of the graph G satisfying (2.1) (see Wu and Chao 2004). For the above node i and for the two trees, T^* and T_{S_n}, both labels identify nodes $j^*, \widehat{j} \in \Gamma_i^{-1}$ that satisfy (2.3).

Therefore

$$\pi^*(i) = l_{j^*i} + \pi^*(j^*) = l_{\widehat{j}i} + \pi^*(\widehat{j})$$

but by (2.4), $l_{\widehat{j}i}$ satisfies $l_{\widehat{j}i} \le l_{ji}, \forall j \in \arg\min_{j\in\Gamma_i^{-1}}\{l_{ji} + \pi^*(j)\}$.

Therefore the length of the arc chosen in T^* cannot be less than the length of the arc in T_{S_n}. This contradiction proves that it cannot exist T^* defined above, and therefore T_{S_n} is a minimum cost shortest-path tree. □

Example 2.2 Let G be the acyclic graph given in Fig. 1. The figure also gives the length of the shortest-path from the source node 0 to each node.

Consider the following choice of the sequence defined by (2.1) on the graph in Fig. 1.

$$S_0 = \{0\}, \qquad S_1 = \{0,1\}, \qquad S_2 = \{0,1,2\},$$
$$S_3 = \{0,1,2,3\}, \qquad S_4 = \{0,1,2,3,4\}, \qquad S_5 = \{0,1,2,3,4,5\},$$
$$S_6 = \{0,1,2,3,4,5,6\}, \qquad S_7 = \{0,1,2,3,4,5,6,7\}.$$

Then, using (2.2), (2.3) and (2.4) we obtain the results described in Table 1.

The sequence of trees $T_{S_i}, i = 1, \ldots, 6$, in Table 1, are minimum cost shortest-path subtrees, and T_{S_7} is the optimal tree. Underlined numbers, in the third column of Table 1, point out the choices made by the algorithm in the construction of the optimal solution in case of ties.

3 A cooperative minimum cost shortest-path tree game

The cooperative minimum cost shortest-path tree game arises when considering the problem of allocating the costs associated with a shortest-path tree in a graph among the agents who are located on the nodes of the graph, except for node 0 that is reserved for a common supplier who does not participate in the cost sharing.

Table 1 Minimum cost shortest-path sub-trees

k	S_k	i_k	$\{l_{ii_k} + \pi^*(i)\}$	$A(i_k)$	$\pi^*(i_k)$
0	{0}	0			0
1	{0, 1}	1	{2 + 0}	(0, 1)	2
2	{0, 1, 2}	2	{2 + 0}	(0, 2)	2
3	{0, 1, 2, 3}	3	{1 + 2}	(1, 3)	3
4	{0, 1, 2, 3, 4}	4	$\{\underline{1} + 3, 2 + 2\}$	(3, 4), (2, 4)	4
5	{0, 1, 2, 3, 4, 5}	5	$\{\underline{1} + 4, 2 + 3\}$	(4, 5), (3; 5)	5
6	{0, 1, 2, 3, 4, 5, 6}	6	{3 + 2}	(2, 6)	5
7	{0, 1, 2, 3, 4, 5, 6, 7}	7	$\{10 + 2, \underline{3} + 5, 3 + 5\}$	(6, 7), (5, 7)	8

We consider the game (N, v), where $N = \{1, \dots, n\}$ is the set of players (agents). Player i, $i = 1, \dots, n$, is associated with node i of the graph $G = (N_0, A)$, and the characteristic function v is defined by:

$$v(S) = c(T_S), \qquad \emptyset \neq S \subseteq N, \qquad v(\emptyset) = 0$$

where T_S is a minimum cost shortest-path tree connecting S with the origin 0 using the arcs in $G = (N_0, A)$. We call this class of cooperative games the Minimum Cost Shortest-Path Tree Games (MCSPTG). The reader may note that in our approach coalitions may use any node in the graph to get a cheapest connection as it has already been done in literature (see Curiel 1997; Fernández et al. 2004).

Let us define the graph $G_{sp} = (N_0, A_{sp})$ as the subgraph of G such that $A_{sp} = \bigcup_{k=1}^{n} A(i_k)$. We call G_{sp} the shortest-paths graph of G. Let $(N, \widehat{v})$ be the standard minimum cost spanning tree game defined on the graph G_{sp}. (Recall that in the standard MCST game arcs outside $S \cup \{0\}$ are not allowed to compute the characteristic function and therefore $\widehat{v}(S) = +\infty$ is allowed.) Then, we consider (N, v^*), the *monotonic cover* of $(N, \widehat{v})$, as the game on the graph G_{sp}, where the characteristic function $v^*(S)$, $S \subset N$, is the cost of a minimum cost spanning tree, $T(S)$, in G_{sp} containing $S \cup \{0\}$ and where arcs outside $S \cup \{0\}$ are allowed (see Curiel 1997). Note that the monotonic cover coincides with the standard MCST game if one assume that arcs outside S can not be used to compute the value of $v^*(S)$. Therefore, this allows us to determine the values of $v(S)$ in the original game in a different way.

Theorem 3.1 *The Minimum Cost Shortest-Path Tree Game (N, v) on the graph G is equivalent to the monotonic cover of the Minimum Cost Spanning Tree Game (N, v^*) on the subgraph G_{sp} of G.*

Proof Let (N, v^*) be the *monotonic cover* of the Minimum Cost Spanning Tree Game on the subgraph G_{sp} of G. So, for each coalition $S \subset N$, $v^*(S)$ is the cost of a minimum cost spanning tree, $T(S)$, in G_{sp} containing $S \cup \{0\}$.

The definition of G_{sp} ensures that $T(S)$ is also a shortest-path tree, because all its arcs satisfy the condition (2.3) since they belong to $\bigcup_{k=1}^{n} A(i_k)$, thus $l_{ij} = \pi^*(j) - \pi^*(i)$. Therefore, the value $v(S)$ of the Minimum Cost Shortest-Path Tree Game on G is the same as the value $v^*(S)$ of the *monotonic cover* of the Minimum Cost Spanning Tree Game on G. □

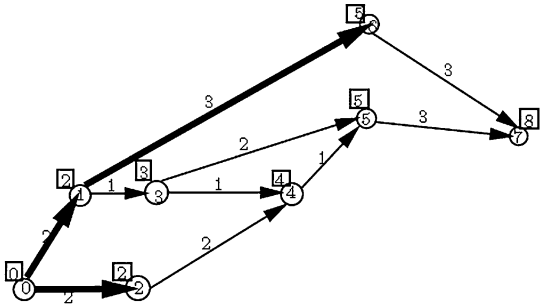

Fig. 2 G_{sp} associated with G, and $T(S)$ for $S = \{2, 6\}$

Example 3.2 This example shows the graph G_{sp} associated with the graph G in Fig. 2.

It clearly follows that for $S = \{2, 6\}$, $v(S) = 7$ using the minimum cost spanning tree that contains $\{0, 2, 6\}$ in the subgraph G_{sp}.

As a consequence of Theorem 3.1, we present some corollaries that illustrate properties for this class of games.

Corollary 3.3 *The game* (N, v) *is monotonic.*

In addition, the core of this game is not empty. Consider the cost allocation that for a given tree, assigns to each node the cost of the incident edge on the unique path that links this node with the root 0 of the tree. In the following, we call this allocation *Birds's cost allocation* according to Bird (1976) and Curiel (1997).

Corollary 3.4 *Bird's cost allocation belongs to the core of the game* (N, v).

Nevertheless, this class of games enjoys the following interesting property.

Theorem 3.5 *Bird's cost allocation is the same allocation for all the minimum cost shortest-path trees of the graph* G.

Proof All the minimum cost shortest-path trees of the graph G are obtained via (2.2), (2.3) and (2.4). In this procedure the values $\pi^*(i)$ for all i do not depend on the optimal subtree, nor the poset (Dusnik and Miller 1941), considered so far.

Thus in a given step, say at node j, we choose the edge according to (2.3), (2.4), and since the value $\pi^*(i)$ for all $i \in \Gamma_j^{-1}$, is independent of the chosen subtree and the poset, the choice of any node $\widehat{i} \in \arg\min_{i\in\Gamma_j^{-1}}\{l_{ij} + \pi^*(i)\}$ satisfying $l_{\widehat{i}j} \leq l_{ij}$, $\forall i \in \arg\min_{\Gamma_j^{-1}}\{l_{ij} + \pi^*(i)\}$, always gives the same value, $l_{\widehat{i}j}$, because otherwise $\pi^*(j)$ would give different values dependent of the poset which is not possible. □

4 A non-cooperative minimum cost shortest-path tree game

In many real life situations players make their decisions independently and therefore it is important to consider the problem from a non-cooperative point of view (see Bergantiños and Lorenzo 2004, 2005; Fernández et al. 2009; Gómez-Rúa and Vidal-Puga 2011).

Now we describe how the players make the connections to the source in a non-cooperative multi-stage game, $\Gamma(G,c)$, associated with each minimum cost shortest-path tree problem in the graph G with costs c.

Initially all players are unconnected. At the first stage, each player makes the decision whether to connect to the source or not. If no player connects or every player connects to the source then the game finishes. Otherwise, the game proceeds to a second stage. In subsequent stages non-connected players face a set of players already connected and have to decide whether to remain unconnected or to connect to one of the connected players or to the source. The game finishes when no more players connect or when all the players are already connected.

In each stage, when player i joins the existing tree, he has to pay the *incremental cost*, i.e., the amount of the cost due to its connection at this stage. When several players decide to join simultaneously, each player has to pay the incremental cost, sequentially, in the numbering of the players (nodes) that is used to identify players with nodes of the graph. Recall that in Sect. 2 (before formula (2.1)), we fixed a numbering of the nodes compatible with the precedence relationship of the graph. Here, we remark that an important property of optimal trees is that any sub-tree of an optimal tree, that connects to the source, must be also an optimal solution for the corresponding subproblem. This key observation implies that in order to produce optimal trees, players *must* use shortest-path to connect to other players and to the source. If at the end of the process, a player is still unconnected, he has to pay a penalty that is much higher than the cost of the shortest-path to the source.

In this game, the decision of each player depends only on the set of players already connected. Denote by $2_0^N(i)$ the set of all the coalitions that contain the source but not player i, $2_0^N(i)=\{S\in 2_0^N \mid i\notin S\}$. Let P_i be the collection of all shortest paths from i to 0. A strategy for player $i\in N$ is a map $x_i: 2_0^N(i)\longrightarrow P_i\cup\{d\}$ such that $x_i(S)=p$ means that player i builds the subpath of p starting from the last node on path p that is a member of S and ending at node i. We assume that no path is called 'd' and reserve this character for the choice of not connecting; $x_i(S)=d$ means that player i does not build any edge when the current tree spans S.

Let $x=(x_i)_{i\in N}$ denote a profile of strategies of the set of players, X_i denote the set of all possible strategies for player i, and X denote the set of all possible profiles of strategies for the entire set of players.

A profile of strategies, x, of the game $\Gamma(G,c)$ induces a graph that is a tree T^x on a subset $S\in 2_0^N$. This tree is a shortest-path spanning tree on S but not necessarily a minimum cost shortest-path spanning tree on S. However, if the total cost of the resulting tree is to be minimized, the solution should consist of a minimum cost shortest-path spanning tree on S.

In what follows, we focus on the characterization of those profiles of strategies that are Nash equilibria and result in minimum cost shortest-path trees on N_0.

To formally define these properties, we introduce the following notation. Given a profile of strategies x for the whole set of players, N, and a subset of players $S\subset N$, denote by x_S (x_{-S}) the projection of x on S ($N\setminus S$) that represents the corresponding profile of strategies for the agents in S ($N\setminus S$). By $(x;x'_S)$ we represent the profile of strategies in which agents in S deviate from x by using the profile of strategies x', that is, $(x;x'_S)_i=x'_i$ for $i\in S$ and $(x;x'_S)_j=x_j$ for all $j\notin S$.

Let $c_i(x)$ denote the connection cost for agent i when a profile of strategies x is adopted. The total cost induced by x is denoted by $c(x)$, $c(x)=\sum_{i\in N}c_i(x)$. (We will slightly abuse of notation by using $c_i(x)$ and $c(x)$ instead of $c_i(T^x)$ and $c(T^x)$.)

Definition 4.1 The profile of strategies $x\in X$ is a *Nash equilibrium* (NE) for the game $\Gamma(G,c)$, if for every agent $i\in N$, $c_i(x)\le c_i(x;x'_i)$ for all $x'_i\in X_i$, $x'_i\neq x_i$.

That is, x is a NE if any deviation of agent i from the profile of strategies x does not yield an improvement in the cost assigned to agent i. The reader may note that in this game, due to the finiteness of the strategy space, there are always Nash equilibrium profiles, at least in mixed strategies.

Unfortunately, a Nash equilibrium strategy does not necessarily result in a tree which is a MCSPT. The following example shows that these two conditions are independent in the sense that there exist strategies that results in a tree which is a MCSPT which are not NE strategies and there are NE strategies that do not yield minimum cost shortest-path trees.

Example 4.2 For the graph in Fig. 1, we consider the joint strategy where each player j chooses to join a tree T_S if there is a player in S such that j can connect to the tree by only one arc. In addition, in this strategy player 3 will connect only if player 4 is in S.

- In the first stage, nodes 1 and 2 are connected.
- In the second stage, nodes 4 and 6 are connected.
- Finally, in the third stage nodes 3, 5 and 7 are connected,

This strategy is a Nash equilibrium but it does not result in a minimum cost spanning tree.

There exist different joint strategies that produce optimal trees but do not result in Nash equilibrium. We consider the joint strategy where player 7 decides to connect to the source in the second stage, using the shortest-path 0, 1, 3, 4, 5 and 7, whereas the remaining players connect to the tree using the shortest length incident arc at the first stage that this is possible.

- As a consequence, in the first stage players 1 and 2 connect to the source, and each one pays two units.
- In the second stage, player 3 connects paying one unit, player 6 connects paying three units, and player 7 connects paying five units, since he uses the nodes in an ascending order.
- In the next stages, players 4 and 5 connect to the tree without any payment.

This strategy produces an optimal tree, but player 7 has a better payoff using different strategies.

The idea underlying the strategies described in the following is that each player would like to connect using his cheapest connection. When each player's cheapest connection generates a shortest-path spanning tree, this tree is a minimum cost shortest-path tree. However, in general this strategy may not yield a shortest-path spanning tree and further analysis is necessary to identify the strategies that the players will adopt in the game.

The necessity of producing minimum cost shortest-path trees forces the players to connect to the source at the stage when the cheapest connection is available to him from among all the shortest-path feasible connections. The rationale of this condition is based on the idea of Bellman-Ford algorithm, see Wu and Chao (2004).

Define $B^i = \{j \in N : (j,i) \in A_i^*,\ p \cup (j,i) \in P_i,\ \forall p \in P_j\}$, $i \in N$, as the set of players j such that player i could connect at its minimum cost to j and the union of (j,i) with any shortest path from the source to j is also a shortest path from the source to i.

Definition 4.3 The strategy $x_i \in X_i$ is a Bellman strategy for player i in the game $\Gamma(G,c)$ if for each $S \in 2_0^N(i)$

$$x_i(S) = \begin{cases} p \cup (j,i) & \text{if } j \in B^i \cap S,\ p \in P_j,\ p \subseteq T_S \\ d & \text{otherwise.} \end{cases}$$

Theorem 4.4 *Bellman strategy profiles are Nash equilibria for the game* Γ *and induce a MCSPT on* N_0.

The proof follows analogously to Theorem 4.3 in Fernández et al. (2009).

Note that each player connects to the tree using this strategy when there is another player in the tree that permits him to connect at his minimum cost shortest path.

Corollary 4.5 *The payoff of any Bellman strategy for the game* $\Gamma(G, c)$ *coincides with the allocation given by the Bird's cost allocation for the MCSPTG.*

The reader may note that one could also have defined the noncooperative game on the reduced graph G_{sp} rather than G. In that case, the *opportune moment strategies*, defined in Fernández et al. (2009), would coincide with the *Bellman strategies* defined above, although clearly each class of strategies are defined on different underlying graphs.

5 Conclusion

Minimum cost spanning tree (MCST) games have been widely considered in the literature of cooperative games because of the importance of the construction cost allocation process. Nevertheless, the type of games considered in this paper has attracted much less attention. In our model, the agents are primarily interested in a shortest path to the root and only secondarily in the cost to be made to build such a path since the root has to be visited on a regular base. This situation gives rise to a different paradigm of cost allocation process in the graph, where the proposed solutions must be different from previous approaches.

We adapt the algorithm in Hansen and Zheng (1996) to the case of acyclic directed graphs to find a minimum cost shortest-path tree in order to be applied to the cost allocation problem associated with a cooperative minimum cost shortest-path tree game. Here, we prove that the cost allocation given by an "à la' Bird" rule provides a core solution in the cooperative game. In addition, we analyze a non-cooperative game based on the connection problem that arises in the above situation. Finally, we found that the strategies that induce the payoff given by the allocation "à la' Bird" are Nash equilibrium profiles.

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